Chapter 3

L^p spaces

3.1 Banach and Hilbert spaces

Definition 3.1. *Let X be a vector space (over $\mathbb{C}$).*

(i) *We call a function $||\cdot|| : X \to [0, \infty)$ a* ***norm*** *if it satisfies*

(a) *(triangle inequality) $||x + y|| \leq ||x|| + ||y||$;*

(b) $||\lambda x|| = |\lambda| \cdot ||x||$ *for any* $\lambda \in \mathbb{C}$*;*

(c) $||x|| = 0$ *iff* $x = 0$*.*

Note: if (a) and (b) hold but (c) is not imposed, then we call $||\cdot||$ a ***seminorm****.*

(ii) *X together with a norm $||\cdot||$ is called a* ***normed space****.*

(iii) *X is called a* ***Banach space*** *if it's a normed space that is complete with respect to the norm: that is, if $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence ($||x_n - x_m|| \to 0$ as $n, m \to \infty$) then $||x_j - x|| \to 0$ for some element $x \in X$.*

(iv) *We call a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ an* ***inner product*** *if it satisfies*

(a) *(conjugate symmetry) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;*

(b) *(linearity in the first argument) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, x \rangle$ for any $\alpha, \beta \in \mathbb{C}$;*

(c) $\langle x, x \rangle \geq 0$ *for all x and* $\langle x, x \rangle = 0$ *iff* $x = 0$*.*

Note: it is not hard to show that $||x|| := \sqrt{\langle x, x \rangle}$ is then a norm.

(v) *X together with an inner product $\langle \cdot, \cdot \rangle$ is called an* ***inner product space*** *(or a pre-Hilbert space).*

(vi) *X is called a* ***Hilbert space*** *if it's an inner product space that is complete with respect to the norm $||x|| := \sqrt{\langle x, x \rangle}$.*

3.2 L^p spaces: definition

Definition 3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. In the following two definition we identiy two functions if they are equal to each other μ-a.e..*

(i) For $1 \le p < \infty$, we define the $L^p(\mu)$ ***space*** *to be the normed space of (equivalence classes of) measurable functions on X such that*

$$||f||_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty$$

(ii) Define the $L^\infty(\mu)$ ***space*** *to be the normed space of (equivalence classes of) measurable functions on X whose* ***essential supremum*** *$||f||_\infty$ (or $\operatorname{ess\,sup} |f(x)|$) is finite:*

$$||f||_\infty := \inf \{a \ge 0 : \mu(\{x : |f(x)| > a\}) = 0\} < \infty$$

Remarks 3.3. (a) That $|| \cdot ||_p$ is in fact a norm (that is, it satisfies the triangle inequality) follows from the Minkowski's inequality, see Section 3.3.

(b) $|| \cdot ||_p$ for $p < 1$ fails the triangle inequality, so L^p isn't a normed space for such p.

(c) In particular, $|f(x)| \le ||f||_\infty$ for μ-a.e. x, and $||f||_\infty$ is the smallest constant with such property.

(d) If X is $\mathbb{N}$, and μ is a counting measure, then it is easy to see that each function in $L^p(\mu)$, $1 \le p \le \infty$, can be identified with the sequence $\{f_j\}_{j=1}^\infty$ (or $\{f_j\}_{j \in \mathbb{Z}}$, respectively) satisfying $\sum_j |f_j|^p < \infty$. This special case of $L^p(\mu)$ is then denoted $\ell^p(\mathbb{N})$. If instead of $\mathbb{N}$, we have any other set A with the counting measure μ, then we also use the notation $\ell^p(A)$ for $L^p(\mu)$.

(e) $\ell^\infty(\mathbb{N})$ is then just the space of all bounded sequences.

3.3 A bunch of inequalities

Definition 3.4. *A function $\phi : (a, b) \to \mathbb{R}$ is called* ***convex*** *if*

$$\phi((1 - \lambda)x + \lambda x) \le (1 - \lambda)\phi(x) + \lambda\phi(y)$$

holds for any $x, y \in (a, b)$ and any $\lambda \in [0, 1]$.

Remarks 3.5. (a) $a = -\infty$ and/or $b = +\infty$ are allowed.

(b) The condition (3.3.2) can be easily rephrased to

$$\frac{\phi(t) - \phi(x)}{t - x} \le \frac{\phi(y) - \phi(t)}{y - t} \tag{3.3.1}$$

for all $a < x < t < y < b$. This can be easily understood geometrically.

Theorem 3.6 (Jensen's Inequality). *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Suppose ϕ is convex on (a, b) and let $f \in L^1(\mu)$ with $f(x) \in (a, b)$ for all $x \in X$. Then*

$$\phi\left(\int_X f \, d\mu \right) \le \int_X \phi(f(x)) \, d\mu \tag{3.3.2}$$

Proof. Let $I = \int f \, d\mu$, $a < I < b$. Let $\beta = \sup_{a<x<I} \frac{\phi(I)-\phi(x)}{I-x}$. Then, see (3.3.1), $\beta \le \frac{\phi(y)-\phi(I)}{y-I}$ for any $I < y < b$. Therefore $\phi(y) \ge \phi(I) + \beta(y - I)$ both for $I < y < b$ as well as $a < y \le I$ (geometrically this is easy to believe too). Since $f(x) \in (a, b)$, we get

$$\phi(f(x)) \ge \phi(I) + \beta(f(x) - I).$$

Then integrating with respect to μ, we get $\int \phi \circ f \, d\mu \ge \phi(I) + 0$. □

Definition 3.7. *$p, q \in [1, \infty]$ are called **conjugate exponents** if*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Examples 3.8. $p = q = 2$ and $p = 1$, $q = \infty$ are the most important special cases.

Theorem 3.9 (**Young's Inequality**). *Suppose p and q are conjugate exponents, $1 < p < \infty$. Then for all $x, y \geq 0$:*

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Proof. Jensen's inequality with $\phi(x) = e^x$, $X = \{x_1, x_2\}$, and $\mu(\{x_1\}) = 1/p$, $\mu(\{x_2\}) = 1/q$, $f(x_1) = p \log x$, $f(x_2) = q \log y$, gives us $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$. □

Theorem 3.10 (**Hölder Inequality**). *Suppose p and q are conjugate exponents, $1 \leq p, q \leq \infty$. If f and g are measurable, then*

$$||fg||_1 \leq ||f||_p \, ||g||_q \tag{3.3.3}$$

Remark 3.11. For $p = q = 2$ this is the **Schwarz inequality** (also, Cauchy–Bunyakovsky in some countries).

Proof. When one of p or q is equal to ∞, the result is obvious. So assume $1 < p < \infty$.

The result is also trivial if one of the norms are 0 or ∞. Note that scalar multiplication preserves the inequality so we may normalize: $F := |f|/||f||_p$ and $G := |g|/||g||_q$.

Apply Young's inequality with $F(x)$ and $G(x)$ instead of x, y:

$$F(x)G(x) \leq \frac{1}{p}F(x)^p + \frac{1}{q}G(x)^q$$

which holds for every x. Integrating, we get

$$\int FG \, d\mu \leq \frac{1}{p} + \frac{1}{q} = 1.$$

□

Theorem 3.12 (**Generalized Hölder's Inequality**). *Let $1 \leq p, q, r \leq \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Then

$$||fg||_r \leq ||f||_p \, ||g||_q \tag{3.3.4}$$

Remark 3.13. This can be generalized even further, see [F, Ex.6.3.31].

Proof. Again, we can assume none of p, q, r are ∞. Then let

$$\tilde{f} = |f|^r, \quad \tilde{g} = |g|^r,$$

and $\tilde{p} = \frac{p}{r}$, $\tilde{q} = \frac{q}{r}$. Then we get $\frac{1}{p} + \frac{1}{q} = 1$, and (3.3.4) becomes reduced to (3.3.3). □

Theorem 3.14 (**Minkowski's Inequality**). *Let $1 \leq p \leq \infty$. Then*

$$||f + g||_p \leq ||f||_p + ||g||_p$$

for any $f, g \in L^p(\mu)$.

Proof. Inequality for $p = 1$ and $p = \infty$ follows from the usual triangle inequality for $\mathbb{C}$.

For $1 < p < \infty$, note that
$$|f + g|^p \le |f|\,|f + g|^{p-1} + |g|\,|f + g|^{p-1}.$$

Then Hölder inequality gives

$$\int |f|\,|f+g|^{p-1} \le \left(\int |f|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q},$$
$$\int |g|\,|f+g|^{p-1} \le \left(\int |g|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q},$$

which, together with $(p-1)q = p$, imply

$$\int |f+g|^p \le (||f||_p + ||g||_p) \left(\int |f+g|^p\right)^{1/q}.$$

Dividing both sides by $\left(\int |f+g|^p\right)^{1/q}$ (assuming it is non-zero) and using $1 - \frac{1}{q} = \frac{1}{p}$, we get, the desired inequality. □

3.4 Completeness

Theorem 3.15. *(i) For any $1 \le p \le \infty$, and any positive measure μ, $L^p(\mu)$ is a Banach space.*

(ii) $L^2(\mu)$ is a Hilbert space with the inner product

$$\langle f, g\rangle := \int_X f(x)\overline{g(x)}\, d\mu(x).$$

Proof. By Minkowski inequality, $||\cdot||_p$ is a norm, so we just need to check completeness.

Let $1 \le p < \infty$ first. Suppose $||f_n - f_m||_p \to 0$ as $n, m \to \infty$. The idea of constructing the limiting function $f(x)$ is to show that the series on the right-hand side of (3.4.1) converges if we choose n_j large enough (so that each term in the series is small).

Indeed, proceeding inductively we get $||f_{n_{j+1}} - f_{n_j}||_p < 2^{-j}$ for some indices $n_1 < n_2 < \ldots$.

Define $g_k = \sum_{j=1}^k |f_{n_{j+1}} - f_{n_j}|$ and $g(x) = \lim_{k\to\infty} g_k(x)$ (exists for all x by monotonicity). By Minkowski $||g_k||_p < 1$ for every k. Since $g_k^p \le g_{k+1}^p$, we can use the Lebesgue Monotone Convergence theorem to conclude that $||g||_p = \lim ||g_k||_p \le 1$. Since $g^p \in L^1(\mu)$, this means that $g(x) < \infty$ for a.e. x. By the definition $g = \sum_{j=1}^\infty |f_{n_{j+1}} - f_{n_j}|$, i.e., the following series also converges (absolutely) for a.e. x:

$$f(x) := f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \tag{3.4.1}$$

(define $f(x) = 0$ on the null-set where the convergence fails). Note that this can also be rewritten as $f(x) = \lim_{j\to\infty} f_{n_j}(x)$ a.e.. Note that $|f| \le |f_{n_1}| + g$ is in L^p since both f_{n_1} and g are in L^p. We need to show that $||f - f_n||_p \to 0$.

Choose $\varepsilon > 0$ and find N such that $||f_n - f_m||_p < \varepsilon$ for all $n, m \ge N$. Then for $m \ge N$, by Fatou's lemma

$$||f - f_m||_p^p = \int \lim_{j\to\infty} |f_{n_j}(x) - f_m(x)|^p d\mu \le \liminf_{j\to\infty} \int |f_{n_j}(x) - f_m(x)|^p d\mu = \liminf_{j\to\infty} ||f_{n_j} - f_m||_p^p \le \varepsilon^p.$$

This shows that $||f - f_m||_p \to 0$ as $m \to \infty$.

Finally, consider the $p = \infty$ case. Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $L^\infty(\mu)$: $||f_n - f_m||_\infty \to 0$ as $n, m \to \infty$. Note that for μ-a.e. x (union of countably many μ-null sets is a μ-null set), $|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$ for all m, n. So $f(x) := \lim f_n(x)$ exists μ-a.e., and we define $f = 0$ for other x.

Let $\varepsilon_m := \sup_{n \geq m} ||f_n - f_m||_\infty$. Since $\varepsilon_m \to 0$ by the Cauchy property, we have $\varepsilon_N \leq 1$ for some large enough N. Then for a.e. x, $|f(x) - f_N(x)| = \lim_{j\to\infty} |f_j(x) - f_N(x)| \leq \lim_{j\to\infty} ||f_j - f_N||_\infty \leq \varepsilon_N < 1$. So $f - f_N \in L^\infty(\mu)$, which implies $f = f + (f_N - f) \in L^\infty(\mu)$. The last inequality also shows that $||f - f_N||_\infty \to 0$ as $N \to \infty$. ☐

3.5 Inclusions for L^p and ℓ^p spaces

Intuition. We want to understand the relationship between L^p spaces for varying p. The idea is that $t^2 \geq t$ (lower exponent is better for convergence) if $t \geq 1$, and $t^2 \leq t$ (higher exponent is better for convergence) if $0 \leq t \leq 1$. We make this rigorous in Theorem 3.16.

Theorem 3.16. *For any $1 \leq p < q < r \leq \infty$, $L^q \subseteq L^p + L^r$, that is any function in $L^q(\mu)$ is the sum of a function in $L^p(\mu)$ and a function in $L^r(\mu)$.*

Proof. Let us split $f \in L^q(\mu)$ into two parts – where $|f| > 1$ and where $|f| \leq 1$: $f = g + h$ with $g = f\,\chi_{\{x:|f|>1\}}$ and $h = f\,\chi_{\{x:|f|\leq 1\}}$. Since $f \in L^q$, we also have $g, h \in L^q$. Now, $|g|^p \leq |g|^q$, so $g \in L^p$, and $|h|^r \leq |h|^q$, so $h \in L^r$ (if $r = \infty$, then $|h| \leq 1$ clearly implies $||h||_\infty \leq 1$). ☐

Theorem 3.17. *For any $1 \leq p < q < r \leq \infty$, $L^p \cap L^r \subseteq L^q$.*

Proof. One can follow the same idea as before: $f = g + h$ with $g = f\,\chi_{\{x:|f|>1\}}$ and $h = f\,\chi_{\{x:|f|\leq 1\}}$. Since $f \in L^p \cap L^r$, we also have $g, h \in L^p \cap L^r$. Now, as before $g \in L^r$ implies $g \in L^q$ (as in the previous proof, since $|g| \geq 1$, we can pass to the lower exponent), and $h \in L^p$ implies $h \in L^q$ (since $|h| \leq 1$, we can pass to the higher exponent). This means $f \in L^q$.

Alternatively, one can prove the inequality

$$||f||_q \leq ||f||_p^{\lambda}\, ||f||_r^{1-\lambda}, \tag{3.5.1}$$

where $\lambda \in (0, 1)$ is defined from $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. This is a direct corollary of generalized Hölder's inequality if we take $|f|^\lambda$ and $|f|^{1-\lambda}$ instead of f and g, and $\frac{p}{\lambda}, \frac{r}{1-\lambda}, q$ instead of p, q, r, respectively, in (3.3.4). ☐

Theorem 3.18. *If $\underline{\mu(X) < \infty}$ and $1 \leq p < q \leq \infty$, then $L^p(\mu) \supseteq L^q(\mu)$.*

Remark 3.19. The inclusion fails if $\mu(X) = \infty$ as a simple counterexample $f(x) \equiv 1$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ shows.

Proof. Note that $\mu(X) < \infty$ means that function 1 is in any L^p. So we only need to worry about functions f on the set $\{x : |f| > 1\}$ and not on $\{x : |f| \leq 1\}$.

Indeed, let $f \in L^q$, and let as before $f = g + h$. Then h is in every L^r $(1 \leq r \leq \infty)$, while $g \in L^q$ implies $g \in L^p$ (we can go to lower exponent). Therefore $f \in L^p$.

Alternatively, one can prove that

$$||f||_p \leq ||f||_q\, \mu(X)^{\frac{1}{p} - \frac{1}{q}}.$$

which follows from Hölder's inequality with functions $|f|^p$ and 1 and exponents $\frac{q}{p}$ and $\frac{q}{q-p}$:

$$||f||_p^p = \int |f|^p \cdot 1\, d\mu \leq ||\, |f|^p \,||_{q/p}\, ||1||_{q/(q-p)} = ||f||_q^p \mu(X)^{(q-p)/q}.$$

☐

Theorem 3.20. *For any $1 \le p < q \le \infty$, we have $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.*

Remark 3.21. One can take any set A instead of $\mathbb{N}$.

Proof. If $f \equiv \{f_j\}_{j=1}^\infty \in \ell^p$ then $\sum_{j=1}^\infty |f_j| < \infty$, so $f_j \to 0$, so eventually $|f_j| < 1$.

Decompose as above $f = g + h$ where $g = f\,\chi_{\{x:|f|>1\}}$ and $h = f\,\chi_{\{x:|f|\le 1\}}$. Then g is supported on finitely many points, so $g \in \ell^r$ for any r. While for h: $h \in \ell^p$ implies that $h \in \ell^q$ (we can go to higher exponent since $|h| \le 1$).

Alternatively, one can prove that for sequences we have

$$||f||_q \le ||f||_p$$

which follows by applying (3.5.1) with $r = \infty$ and combining it with $||f||_\infty \le ||f||_p$. □

3.6 Dense subspaces of L^p spaces

Intuition. Given a function in $L^p(\mu)$ space, it is natural to ask how well we can approximate it by a simpler class of functions, such as simple or continuous functions. We explore these questions here.

Theorem 3.22. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

Let S be the class of (complex-valued) simple measurable functions $\sum_{j=1}^n \alpha_j \chi_{E_j}$, where $n < \infty$, $\alpha_j \in \mathbb{C}$, $\mu(E_j) < \infty$.

Let $\tilde{S}$ be the class of (complex-valued) simple measurable functions $\sum_{j=1}^n \alpha_j \chi_{E_j}$, where $n < \infty$, $\alpha_j \in \mathbb{C}$, but with $\mu(E_j)$ allowed to be infinite.

(i) S is dense in $L^p(\mu)$ for any $1 \le p < \infty$.

(ii) $\tilde{S}$ is dense in $L^\infty(\mu)$.

Remark 3.23. In general (i) wouldn't work for $p = \infty$, as the counterexample $f \equiv 1$ on $L^\infty(\mathbb{R}, m)$ shows.

Proof. (i) Clearly, $S \subset L^p$. Now, given $f \in L^p \cap L^+$, approximate f from below by simple functions ϕ_n as usual (see the proof of Proposition 2.8). Then $0 \le \phi_n \le f$, $\phi_n \nearrow f$. Note that $\phi_n \le f$, so $\phi_n \in L^p$, so $\mu(E_j) < \infty$ for any ϕ_n. Since $|f - \phi_n|^p \le |f|^p$, we can use Dominated Convergence Theorem to conclude that $\lim ||f - \phi_n||_p = \lim(\int |f - \phi_n|^p d\mu)^{1/p} = 0$, in other words, f is in the closure of S. For complex f, we approximate $\operatorname{Re} f$ and $\operatorname{Im} f$ separately.

(ii) For $f \in L^\infty(\mu) \cap L^+$, first we choose a representative of the equivalence class of f that is bounded. Then we use again the approximation $\{\phi_n\}$ from the proof of Proposition 2.8. Clearly, $\phi_n \in \tilde{S}$ and $||\phi_n - f||_\infty \le \frac{1}{2^n}$ for n large enough. □

Theorem 3.24. *Let $(X, \mathcal{M}, \mu)$ be a measure space with X locally compact and Hausdorff. Suppose μ is regular, Borel, σ-finite. Let $1 \le p < \infty$.*

Then $C_c(X)$ is dense in $L^p(\mu)$.

Remark 3.25. It is clear that for $p = \infty$ this fails in general. For example, if $X = \mathbb{R}^n$, $\mu = m^n$, then the completion of $C_c(\mathbb{R}^n)$ in the $||\cdot||_\infty$-norm is not L^∞ but $C_0(\mathbb{R}^n)$, the space of all continuous functions on $\mathbb{R}^n$ which vanish at ∞, that is, those f for which $\lim_{|x|\to\infty} f(x) = 0$. This can be generalized to more general setting than $X = \mathbb{R}^n$.

Proof. By the previous theorem, we just need to be able to $||\cdot||_p$-approximate functions χ_E with $\mu(E) < \infty$ by $C_c(X)$ functions.

Given $\varepsilon > 0$, by regularity and σ-finiteness of μ (see Theorem 1.35 — we can choose compact rather than just closed by using inner regularity; this works even for σ-finite case as countable intersection of compacts is compact for Hausdorff spaces), we can find a compact set $K \subseteq E$ and an open set $U \supseteq E$ such that $\mu(U \setminus K) < \varepsilon$. By Urysohn's lemma applied to the closed sets K and U^c, we can find a function $f \in C_c(X)$ such that $\chi_K \leq f \leq \chi_U$. Then $||\chi_E - f||_p \leq \mu(U \setminus K)^{1/p} < \varepsilon^{1/p}$. □

3.7 Linear functionals

Recall that in Section 2.24 we had discussed linear functionals on the space $C_c(X)$ of continuous compactly supported functions. Linear functionals can of course be defined over arbitrary vector spaces.

Definition 3.26. *Let X be a vector space over $\mathbb{C}$.*

*(i) We say that a map $\phi : X \to \mathbb{C}$ is a **linear functional on** X if $\phi(x+y) = \phi(x)+\phi(y)$ and $\phi(\alpha x) = \alpha\phi(x)$ for $\alpha \in \mathbb{C}$.*

*(ii) The space of all linear functionals on X forms a vector space which is called the **algebraic dual space of** X.*

Remark 3.27. If X is equipped with a partial order $\leq$ that is compatible with the vector addition and scalar multiplication (in the natural way you'd expect), then we call a functional positive if $x \geq 0$ implies $\phi(x) \geq 0$. We encountered this in Section 2.24 in the special case when X was the (partially ordered) space of continuous compactly supported functions.

Definition 3.28. *Now let X be a Banach space with norm $||\cdot||$.*

*(i) We say that a linear functional ϕ is **bounded** (or **continuous**) if there is $C > 0$ such that $|\phi(x)| \leq C\,||x||$ for all $x \in X$.*

*(ii) The space of all bounded linear functionals on X forms a vector space which is called the **dual space of** X, denoted by X^*.*

Remarks 3.29. (a) Some authors may call X^* the *continuous dual space* or *topological dual space.* We will just call it dual.

(b) Clearly, X^*, the dual space of X, is a subspace of the algebraic dual space of X.

(c) If X is a Banach space, then X^* is easily seen to be a normed space with the norm defined by

$$||\phi|| = \sup\{|\phi(x)| : x \in X, ||x|| \leq 1\} = \sup\left\{\frac{|\phi(x)|}{||x||} : x \in X, x \neq 0\right\}.$$

In fact, it is not much work to show that X^* is complete, i.e., a Banach space.

(d) A rough way to state the Remark 2.76(c) (which is also referred to as a Riesz–Markov representation theorem) is to say that the dual $C_0(X)^*$ of $C_0(X)$ (space of continuous functions vanishing at infinity, the completion of $C_c(X)$) is the space of all (complex, in particular finite) regular Borel measures on X.

3.8 Duals of L^p

Intuition. Given $g \in L^q(\mu)$ $(1 \le q \le \infty)$, according to Hölder's inequality, the map $f \mapsto \int fg\, d\mu$ is a bounded linear functional on $L^p(\mu)$. Does every bounded linear functional on L^p arise in this way? The answer is yes for $1 \le p < \infty$ (at least if μ is σ-finite), but not for $p = \infty$.

Theorem 3.30. *Suppose $1 \le p < \infty$ and μ is a σ-finite (positive) measure. Then for any bounded linear functional ϕ on $L^p(\mu)$ there is a unique $g \in L^q(\mu)$ (where $\frac{1}{p} + \frac{1}{q} = 1$) such that*

$$\phi(f) = \int fg\, d\mu, \qquad \text{for all } f \in L^p(\mu). \tag{3.8.1}$$

Moreover, $||\phi|| = ||g||_q$.

Remarks 3.31. (a) Morally, one can say that $(L^p)^* = L^q$ if $1 \le p < \infty$ and μ is σ-finite.

(b) The statement of the theorem is still correct without the σ-finiteness assumption provided that $1 < p < \infty$.

(c) In particular, $(L^p)^{**} = L^p$ for $1 < p < \infty$. Spaces satisfying such condition are called reflexive.

(d) The statement for $p = \infty$ fails. Indeed, $(L^\infty)^*$ is much bigger than L^1.

Proof. Uniqueness: if g and $\tilde{g}$ both satisfy (3.8.1) then we can take $f = \chi_E$ for any measurable E with $\mu(E) < \infty$, giving $\int_E (g - \tilde{g})\, d\mu = 0$. This implies $g - \tilde{g} = 0$ a.e. using σ-finiteness of μ.

We will suppose that $\mu(X) < \infty$ and leave the extension to the σ-finite case as an exercise ([F190]). Note that (3.8.1) for $f = \chi_E$ takes the form $\phi(\chi_E) = \int_E g\, d\mu$, which looks like a (complex) μ-absolutely continuous measure. This motivates us to define

$$\nu(E) = \phi(\chi_E), \qquad \text{for any } E \in \mathcal{M}.$$

We want to show that $\nu(E) = \int_X \chi_E g\, d\mu = \int_E g\, d\mu$ for some $g \in L^q$. To do this, we will show: (1) ν is a (complex) measure; (2) ν is μ-a.c.; (3) equality (3.8.1) holds with $g := \frac{d\nu}{d\mu} \in L^1$; (4) g is in $L^q(\mu)$ and $||g||_q = ||\phi||$.

(1) Finite-additiveness of ν follows from linearity of ϕ and $\chi_{A \cup B} = \chi_A + \chi_B$ for disjoint sets A and B. For σ-additivity, let $E_\infty = \coprod_{j=1}^\infty A_j$. Define $E_n = \coprod_{j=1}^n A_j$. We need to show $\nu(E_\infty)$ is equal to $\sum_{j=1}^\infty \nu(A_j) \equiv \lim \nu(E_n)$. We use continuity of ϕ to get $|\nu(E_\infty) - \nu(E_n)| = |\phi(\chi_{E_\infty} - \chi_{E_n})| \le C||\chi_{E_\infty} - \chi_{E_n}||_p$. Now note that$||\chi_{E_\infty} - \chi_{E_n}||_p = (\mu(E_\infty \setminus E_n))^{1/p} \to 0$ by continuity of μ. (recall that $p < \infty$).

(2) If $\mu(E) = 0$, then $\chi_E(x) = 0$ (μ-a.e.), so that $||\chi_E||_p = 0$, which implies $\nu(E) = \phi(\chi_E) = 0$ by linearity. Therefore $\nu \ll \mu$.

(3) By (2) and the Radon–Nikodym theorem, $d\nu = g\, d\mu$ for some $g \in L^1(\mu)$. In other words,

$$\int_E d\nu = \phi_{\chi_E} = \int_E g\, d\mu = \int_X \chi_E g\, d\mu.$$

By linearity of integral and of ϕ, we get (3.8.1) for any f that is a simple function.

We can further extend (3.8.1) to $f \in L^\infty$: indeed, by Theorem 3.22, we can find simple functions $s_n \to f$ in $|| \cdot ||_\infty$-norm, which implies $s_n \to f$ in $|| \cdot ||_p$-norm since $\mu(X) < \infty$, and then we can take limits of both sides in (3.8.1) with s_n. We will get (3.8.1) with $f \in L^p(\mu)$ later; having $f \in L^\infty(\mu)$ will be sufficient for now.

(4) Suppose first that $1 < p < \infty$ (so that $p \ne 1$, $q \ne \infty$). Then define $f = |g|^{q-1}\overline{\operatorname{sgn} g}$. Note that $|f|^p = |g|^q = fg$, so we expect from (3.8.1)

$$\int_X |g|^q\, d\mu = \int fg\, d\mu = \phi(f) \le ||\phi||\, ||f||_p = ||\phi|| \left(\int_X |g|^q\, d\mu \right)^{1/p},$$

but we cannot plug f into (3.8.1) since we don't have $f \in L^\infty$. To fix this, let $f_n = |g|^{q-1}\overline{\operatorname{sgn} g}\chi_{E_n}$ where $E_n = \{x : |g(x)| \le n\}$. Then $|f_n|^p = |g|^q = fg$ on E_n, $f_n \in L^\infty$, and we get

$$\int_{E_n} |g|^q \, d\mu = \int_X f_n g \, d\mu = \phi(f_n) \le ||\phi|| \, ||f_n||_p = ||\phi|| \left(\int_{E_n} |g|^q \, d\mu \right)^{1/p},$$

which implies $\left(\int \chi_{E_n} |g|^q \, d\mu\right)^{1/q} \le ||\phi||$. Applying Monotone Convergence Theorem, we get $g \in L^q$ and $||g||_q \le ||\phi||$. This allows us to extend (3.8.1) to $f \in L^p(\mu)$ in the exact same way as before: for any $f \in L^p$, take simple functions $s_n \to f$ in $||\cdot||_p$-norm, and then take limits of both sides of (3.8.1). Because $g \in L^q$, this works now. Finally, having (3.8.1) for all $f \in L^p$ allows us to use Hölder's inequality to conclude $||\phi|| \le ||g||_q$, so we get $||\phi|| = ||g||_q$.

Now let $p = 1$, $q = \infty$. Take any $M < ||g||_\infty$, and let $A = \{x : |g(x)| > M\}$. Note that $0 < \mu(A) < \infty$, and we can take $f = \chi_A \overline{\operatorname{sgn} g}$. Since $f \in L^\infty$, (3.8.1) can be applied to get

$$M\mu(A) \le \int_A |g| \, d\mu = \int_X fg \, d\mu = \phi(f) \le ||\phi|| \, ||f||_1 = ||\phi|| \, \mu(A).$$

so we proved that $M < ||g||_\infty$ implies $M \le ||\phi||$. This proves that $||g||_\infty \le ||\phi||$ and in particular $g \in L^\infty$. This allows to extend (3.8.1) to all $f \in L^1$, and then use Hölder's inequality to conclude $||\phi|| \le ||g||_\infty$, so that $||\phi|| = ||g||_q$. ☐

3.9 Riesz Representation Theorem

Intuition. The duality theorem from the previous section states in particular that $(L^2)^* = L^2$, or more precisely, every bounded linear functional on $L^2(\mu)$ has the form

$$\phi(f) = \int fg \, d\mu, \qquad \text{for all } f \in L^2(\mu) \tag{3.9.1}$$

for some $g \in L^2$. The last expression can also be written as $\phi(f) = \langle f, \bar{g} \rangle$. This is the special case $H = L^2(\mu)$ of the Riesz Representation Theorem which holds for an arbitrary Hilbert space H.

Theorem 3.32 (Riesz Representation Theorem)**.** *Let H be a Hilbert space. For any $g \in H$, define*

$$\phi_g(f) = \langle f, g \rangle, \qquad \text{for any } f \in H. \tag{3.9.2}$$

Then $\phi_g \in H^*$, and conversely, every bounded linear functional on H has the form ϕ_g for a unique $g \in H$.

Proof. [F187]

The uniqueness of g: if $\langle f, g \rangle = \langle f, \tilde{g} \rangle$ for all $f \in H$. then taking $f = g - \tilde{g}$, we get $||g - \tilde{g}|| = 0$, i.e., $g = \tilde{g}$.

Now for existence, if ϕ is a bounded linear functional on H, then either $\phi \equiv 0$ (in which case we take $g = 0$), or otherwise, let $K = \ker \phi = \{f \in H : \phi(f) = 0\}$. Note that in order for (3.9.2) to hold, we must have $g \in K^\perp$. Choose any $z \in K^\perp$ (K is a closed proper subspace in H, so $K^\perp \ne \{0\}$). Then for arbitrary f, $\phi(f)z - \phi(z)f$ is in K, so

$$0 = \langle \phi(f)z - \phi(z)f, z \rangle = \phi(f)||z||^2 - \phi(z) \langle f, z \rangle.$$

Rearranging we get

$$\phi(f) = \left\langle f, \frac{\overline{\phi(z)}z}{||z||^2} \right\rangle,$$

so we can take $q = \frac{\overline{\phi(z)}z}{||z||^2}$. ☐

3.10 Linear operators: definition

Definition 3.33. *Let X and Y be normed spaces.*

(i) We say that a function $T : X \to Y$ is a ***bounded linear operator*** *if T is linear and there exists $C > 0$ such that $||T(x)||_Y \leq C\,||x||_X$ for all $x \in X$.*

(ii) The ***operator norm*** *$||T||$ of a bounded linear operator T is defined to be the smallest such constant C, or, in other words:*

$$||T|| = \sup\{||T(x)||_Y : x \in X, ||x||_X \leq 1\} = \sup\left\{\frac{||T(x)||_Y}{||x||_X} : x \in X, x \neq 0\right\}.$$

(iii) The space of all bounded linear operators from X to Y is denoted by $L(X, Y)$.

Remarks 3.34. 1. In particular, $L(X, \mathbb{C}) = X^*$.
2. It can be shown that if Y is Banach, then $L(X, Y)$ with the operator norm is also a Banach space.

3.11 Riesz–Thorin Interpolation Theorem

Theorem 3.35 (Riesz–Thorin Interpolation Theorem)**.** *Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are two (σ-finite) measure spaces, and let $p_0, p_1, q_0, q_1 \in [1, \infty]$. For each $0 < t < 1$, let p_t, q_t be defined through*

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \qquad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

Let $T : L^{p_0}(\mu) + L^{p_1}(\mu) \to L^{q_0}(\nu) + L^{q_1}(\nu)$ be linear and satisfy

$$||T(f)||_{q_0} \leq M_0 ||f||_{p_0},$$
$$||T(f)||_{q_1} \leq M_1 ||f||_{p_1}.$$

Then for any $0 \leq t \leq 1$, T is a bounded linear operator from $L^{p_t}(\mu)$ to $L^{q_t}(\nu)$ and $||T(f)||_{q_t} \leq M_0^{1-t} M_1^t ||f||_{p_t}$.

Proof. This is mostly complex analysis and will be skipped, see [F200–202] if interested. □
